

A Picard family of curves and hypergeometric functions over finite fields I

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Abstract

We give an expression for the trace of Frobenius for the family of curves

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

over finite fields in terms of finite field hypergeometric functions.

1 Introduction

Let p be an odd prime and \mathbb{F}_p be the finite fields with p elements. For $\lambda \in \mathbb{F}_p \setminus \{0, 1\}$ we define an elliptic curve over \mathbb{F}_p in the Legendre family by

$$E : y^2 = x(x-1)(x-\lambda)$$

Koike [4] showed that, for all odd primes p , the trace of Frobenius for curves in this family can be expressed in terms of Greene's finite field hypergeometric function.

$$a_p(E) = -p\phi(-1)_2 F_1 \left(\begin{matrix} \phi & \phi \\ \varepsilon \end{matrix}; \lambda \right)_p$$

where ε is the trivial character and ϕ is a quadratic character of \mathbb{F}_p^\times .

Let q be a power of a rational prime p and \mathbb{F}_q be the finite fields with q elements. For $\lambda, \mu \in \mathbb{F}_q$ with $\lambda\mu(\lambda-1)(\mu-1) \neq 0$ we define a smooth projective curve over \mathbb{F}_q in the Picard family by

$$C : y^3 = x(x-1)(x-\lambda)(x-\mu).$$

This is a genus 3 curve.

We show that the trace of Frobenius $a_q(C)$ for C can be expressed in terms of finite field Appell hypergeometric functions.

Theorem 1.1. *Let $q = p^e$ be a power of prime such that $q \equiv 1 \pmod{3}$, χ_3 be a cubic character and ε be the trivial character of \mathbb{F}_q , we have*

$$a_q(C) = -q \sum_{j=1}^2 \chi_3^{2j}(-1) F_1 \left(\begin{matrix} \chi_3^j & \chi_3^j & \chi_3^j \\ \varepsilon \end{matrix}; \lambda \mu \right)_q.$$

2 Preliminaries

2.1 Finite field hypergeometric functions

Recall that the hypergeometric series is defined by

$${}_2F_1\left(\begin{matrix} a & b \\ c \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(1, n)(c, n)} x^n$$

where

$$(a, n) = \begin{cases} a(a+1) \cdots (a+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

In [2], Greene defined a finite field analogue of classical hypergeometric series. Let $q = p^a$ a power of prime, \mathbb{F}_q be a finite field of q elements. For a character $\chi \in \widehat{\mathbb{F}_q^\times}$, we extend it to all of \mathbb{F}_q by setting

$$\chi(0) = \begin{cases} 0, & \chi \neq \varepsilon \\ 1, & \chi = \varepsilon. \end{cases}$$

where ε is the trivial character.

For two characters A, B of \mathbb{F}_q we define the normalized Jacobi sum

$$\left(\frac{A}{B}\right) = \frac{B(-1)}{q} J(A, \overline{B})$$

where $J(A, B) = \sum_{t \in \mathbb{F}_q} A(t)B(1-t)$ is the usual Jacobi sum. The following properties of normalized Jacobi sum are found in [2].

Lemma 2.1. *Let A, B be characters of \mathbb{F}_q we have*

(1)

$$\overline{A}(1-t) = \delta(t) + \frac{q}{q-1} \sum_{\chi \in \widehat{k}} \begin{pmatrix} A\chi \\ \chi \end{pmatrix} \chi(t)$$

(2)

$$J(A, B) = qB(-1) \left(\frac{A}{\overline{B}}\right)$$

(3)

$$\left(\frac{A}{B}\right) = \left(\frac{A}{A\overline{B}}\right)$$

where

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

for $t \in \mathbb{F}_q$.

For three characters $A, B, C \in \widehat{\mathbb{F}_q^\times}$ Greene defined the finite field hypergeometric function ${}_2F_1$ by

$${}_2F_1\left(\begin{matrix} A & B \\ C \end{matrix}; x\right)_q = \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{B\chi}{C\chi} \chi(x)$$

For four characters $A, B_1, B_2, C \in \widehat{\mathbb{F}_q^\times}$ Ghosh defined the finite field hypergeometric function of two variables F_1 by

$$F_1\left(\begin{matrix} A & B_1 & B_2 \\ C \end{matrix}; x \ y\right)_q = \frac{q^2}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{C\chi} \binom{B_1\chi_1}{\chi_1} \binom{B_2\chi_2}{\chi_2} \chi_1(x) \chi_2(y)$$

where $\chi = \chi_1 \chi_2$.

This function is a finite field analogue of Appell hypergeometric series F_1 .

$$F_1\left(\begin{matrix} a & b_1 & b_2 \\ c \end{matrix}; x \ y\right) = \sum_{m, n \geq 0} \frac{(a, m+n)(b_1, m)(b_2, n)}{(1, m)(1, n)(c, m+n)} x^m y^n.$$

The next lemma is proved by Ghosh [3].

Lemma 2.2. (*Ghosh*)

$$F_1\left(\begin{matrix} A & B_1 & B_2 \\ C \end{matrix}; x \ y\right)_q = \varepsilon(xy) \frac{AC(-1)}{q} \sum_{t \in \mathbb{F}_q} A(t) \overline{AC}(1-t) \overline{B_1}(1-xt) \overline{B_2}(1-yt)$$

where $\chi = \chi_1 \chi_2$.

Proof. Suppose

$$g(x, y) = \varepsilon(xy) \frac{AC(-1)}{p} \sum_{t \in \mathbb{F}_q} A(t) \overline{AC}(1-t) \overline{B_1}(1-xt) \overline{B_2}(1-yt)$$

and we consider the cases in $xy \neq 0$.

$$\begin{aligned} \overline{B_1}(1-xt) \overline{B_2}(1-yt) &= \frac{q^2}{(q-1)^2} \left(\sum_{\chi_1 \in \widehat{\mathbb{F}_q^\times}} \binom{B_1\chi_1}{\chi_1} \chi_1(xt) \right) \left(\sum_{\chi_2 \in \widehat{\mathbb{F}_q^\times}} \binom{B_2\chi_2}{\chi_2} \chi_2(yt) \right) \\ &= \frac{q^2}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \binom{B_1\chi_1}{\chi_1} \binom{B_2\chi_2}{\chi_2} \chi_1(xt) \chi_2(yt) \\ &= \frac{q^2}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \binom{B_1\chi_1}{\chi_1} \binom{B_2\chi_2}{\chi_2} \chi(t) \chi_1(x) \chi_2(y) \end{aligned}$$

By (2.1) we have

$$\begin{aligned}
g(x, y) &= \frac{qAC(-1)}{(q-1)^2} \sum_{t \in \mathbb{F}_q} A(t) \overline{AC}(1-t) \sum_{\chi_1, \chi_2 \in \widehat{k}} \begin{pmatrix} B_1 \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} B_2 \chi_2 \\ \chi_2 \end{pmatrix} \chi(t) \chi_1(x) \chi_2(y) \\
&= \frac{qAC(-1)}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} B_1 \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} B_2 \chi_2 \\ \chi_2 \end{pmatrix} \chi_1(x) \chi_2(y) \sum_{t \in \mathbb{F}_q} A\chi(t) \overline{AC}(1-t) \\
&= \frac{qAC(-1)}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} B_1 \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} B_2 \chi_2 \\ \chi_2 \end{pmatrix} \chi_1(x) \chi_2(y) J(A\chi, \overline{AC}) \\
&= \frac{qAC(-1)}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} B_1 \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} B_2 \chi_2 \\ \chi_2 \end{pmatrix} \chi_1(x) \chi_2(y) q \overline{AC}(-1) \begin{pmatrix} A\chi \\ \overline{AC} \end{pmatrix} \\
&= \frac{q^2}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} A\chi \\ \overline{AC} \end{pmatrix} \begin{pmatrix} B_1 \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} B_2 \chi_2 \\ \chi_2 \end{pmatrix} \chi_1(x) \chi_2(y) \\
&= \frac{q^2}{(q-1)^2} \sum_{\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} A\chi \\ C\chi \end{pmatrix} \begin{pmatrix} B_1 \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} B_2 \chi_2 \\ \chi_2 \end{pmatrix} \chi_1(x) \chi_2(y)
\end{aligned}$$

□

2.2 Picard curves over finite fields

Let $q = p^a$ a power of prime $p > 3$, \mathbb{F}_q be a finite field of q elements and let C be a smooth projective curve of genus 3 over \mathbb{F}_q with an affine model

$$y^3 = x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$

here it is supposed $f(x) = x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0$ has no multiple root. It is called a Picard curve.

For $\lambda, \mu \in \mathbb{F}_q$ with $\lambda\mu(\lambda-1)(\mu-1) \neq 0$ we define a Picard curve over \mathbb{F}_q by

$$C : y^3 = x(x-1)(x-\lambda)(x-\mu)$$

This is the Picard family of curves.

Let $\sharp C(\mathbb{F}_q)$ be the number of \mathbb{F}_q -rational points of C and we let

$$a_q(C) = 1 + q - \sharp C(\mathbb{F}_q).$$

$a_q(C)$ is called the trace of Frobenius for C .

The number of \mathbb{F}_q -rational points of C can be expressed in terms of characters of \mathbb{F}_q^\times .

$$\sharp C(\mathbb{F}_q) = 1 + \sum_{\chi^3 = \varepsilon} \chi(x(x-1)(x-\lambda)(x-\mu))$$

Let χ_3 be a cubic character of \mathbb{F}_q^\times we have

$$\begin{aligned}\#C(\mathbb{F}_q) &= 1 + \sum_{x \in \mathbb{F}_q} \sum_{j=0}^2 \chi_3^j(x(x-1)(x-\lambda)(x-\mu)) \\ &= 1 + q + \sum_{x \in \mathbb{F}_q} \sum_{j=1}^2 \chi_3^j(x(x-1)(x-\lambda)(x-\mu))\end{aligned}$$

2.3 Proof of main theorem

Lemma 2.3. *Let A, B_1, B_2, C be characters of \mathbb{F}_q and $x, y \in \mathbb{F}_q$ we have*

$$F_1\left(\begin{matrix} A & B_1 & B_2 \\ & C & \end{matrix}; x \ y\right)_q = \varepsilon(xy) \frac{AC(-1)}{p} \sum B_1 B_2 \overline{C}(s) \overline{AC}(s-1) \overline{B}_1(s-x) \overline{B}_2(s-y)$$

Proof. By (2.2)

$$F_1\left(\begin{matrix} A & B_1 & B_2 \\ & C & \end{matrix}; x \ y\right)_q = \varepsilon(xy) \frac{AC(-1)}{q} \sum_t A(t) \overline{AC}(1-t) \overline{B}_1(1-xt) \overline{B}_2(1-yt).$$

Suppose $xy \neq 0$ and set $s = t^{-1}$,

$$\begin{aligned}F_1\left(\begin{matrix} A & B_1 & B_2 \\ & C & \end{matrix}; x \ y\right)_q &= \frac{AC(-1)}{q} \sum A(t) \overline{AC}(1-t) \overline{B}_1(1-xt) \overline{B}_2(1-yt) \\ &= \frac{AC(-1)}{q} \sum A\left(\frac{1}{s}\right) \overline{AC}\left(\frac{s-1}{s}\right) \overline{B}_1\left(\frac{s-x}{s}\right) \overline{B}_2\left(\frac{s-y}{s}\right) \\ &= \frac{AC(-1)}{q} \sum B_1 B_2 \overline{C}(s) \overline{AC}(s-1) \overline{B}_1(s-x) \overline{B}_2(s-y)\end{aligned}$$

□

Proof of Theorem 1.1. The trace of Frobenius for C is given by

$$\begin{aligned}a_q(C) &= 1 + q - \#C(\mathbb{F}_q) \\ &= - \sum_{x \in \mathbb{F}_q} \chi_3(x(x-1)(x-\lambda)(x-\mu)) \\ &\quad - \sum_{x \in \mathbb{F}_q} \chi_3^2(x(x-1)(x-\lambda)(x-\mu)).\end{aligned}$$

Let $A = B_1 = B_2 = \chi_3$, $C = \varepsilon$ the hypergeometric function has the form

$$\begin{aligned}F_1\left(\begin{matrix} \chi_3 & \chi_3 & \chi_3 \\ & \varepsilon & \end{matrix}; x \ y\right)_q &= \frac{\chi_3(-1)}{q} \sum \chi_3^2(s) \overline{\chi_3}(s-1) \overline{\chi_3}(s-x) \overline{\chi_3}(s-y) \\ &= \frac{\chi_3(-1)}{q} \sum \overline{\chi_3}(s(s-1)(s-x)(s-y)).\end{aligned}$$

We get

$$\sum_{t \in \mathbb{F}_q} \chi_3^2(t(t-1)(t-x)(t-y)) = q\chi_3^2(-1)F_1\left(\begin{matrix} \chi_3 & \chi_3 & \chi_3 \\ \varepsilon \end{matrix}; x \ y\right)_q.$$

Similarly, let $A = B_1 = B_2 = \chi_3^2$, $C = \varepsilon$

$$\sum_{t \in \mathbb{F}_q} \chi_3^2(t(t-1)(t-x)(t-y)) = q\chi_3^2(-1)F_1\left(\begin{matrix} \chi_3 & \chi_3 & \chi_3 \\ \varepsilon \end{matrix}; x \ y\right)_q.$$

□

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